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Plane Frame Functions and Pure States in Hilbert Space

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Abstract

An elementary proof of Gleason's theorem for pure states in a Hilbert space is given. This theorem follows from a general result concerning plane frame functions in real inner product spaces.

1. *Introduction*

Gleason's theorem (Gleason, 1957) for states in a Hilbert space is a cornerstone of the mathematical foundations of quantum mechanics and is one of the most profound theorems that have been proved in this subject so far (Jauch, 1968; Mackey, 1963; Varadarajan, 1968). The proof is fairly long and complicated, involving such notions as representations of the rotation group together with intricate, delicate arguments in spherical geometry. Since the time of Gleason's proof (1957) mathematicians and physicists have sought to find simpler proofs (Brown, 1968) and generalizations for example to states on von Neumann algebras. Of course, one of the hopes in finding simpler proofs is that generalizations may then be easier to discover.

In this paper we give an elementary proof of Gleason's theorem for pure states. Although this does not give Gleason's full theorem it does give an important part of the theorem. Since physicists are usually concerned with pure states this part of Gleason's theorem is extremely important as far as applications are concerned. The proof is self-contained and elementary to the extent that it requires only a rudimentary knowledge of real analysis. We first prove a quite general lemma and a theorem which seem to have interest in their own right. Gleason's theorem for pure states then follows as a simple corollary. Many of the ideas of the following proofs may be found in Piron's survey (Piron, 1970) where a geometrical argument is given. In the following proofs purely analytic arguments are given which may produce new insights to the problem.

2. *Plane Frame Functions and States*

Let X be a real inner product space with inner product $\langle\cdot,\cdot\rangle$ and norm $|x| = \langle x,x\rangle^{1/2}$. If $x, y \in X$ are orthogonal we write $x \perp y$. We denote the real line by R and the non-negative real line by R^+.

Lemma 1. Let $\dim X \geqslant 2$ and $f: X \to R$ satisfies $f(x+y) \leqslant f(y)$ whenever $x \perp y$. Then f is decreasing in the sense that $|x| < |y|$ implies $f(y) \leqslant f(x)$ and f is continuous on a dense subset of X.

Proof. We first show that $f(\alpha y) \leqslant f(y)$ for $\alpha \geqslant 1$. Let x satisfy $|x| = 1$ and $x \perp y$ and let $z = (\alpha - 1)y - (\alpha - 1)^{1/2}|y|x$. Then

$$z \perp y + (\alpha - 1)^{1/2}|y|x$$

so that

$$f(\alpha y) = f(y + (\alpha - 1)^{1/2}|y|x + z) \leqslant f(y + (\alpha - 1)^{1/2}|y|x) \leqslant f(y)$$

Clearly $f(y) \leqslant f(0)$. Now suppose $|y| > |x| > 0$ and let M be the two-dimensional subspace generated by x and y. Let $n > 4$ be a positive integer and let $x_1 \in M$ satisfy $x_1 \perp x$, $|x_1| = |x| \tan \pi/n$. Let $x_2 \in M$ satisfy $x_2 \perp x + x_1$, $\langle x_2, x_1\rangle \geqslant 0$ and $|x_2| = |x + x_1| \tan \pi/n$. Continuing in this way we get $2n$ vectors:

$$x, y_1 = x + x_1, \ldots, y_{2n-1} = x + x_1 + \ldots + x_{2n-1}$$

Connecting these points we obtain a plane $2n$-sided polygon P_n. Now

$$|y_1| = |x|(\cos \pi/n)^{-1}, \ldots, |y_n| = |x|(\cos \pi/n)^{-n}$$
$$|y_{n+1}| = |x|(\cos \pi/n)^{-n+1}, \ldots, |y_{2n-1}| = |x|(\cos \pi/n)^{-1}$$

It is thus clear that y_n is the point on P_n with largest norm. Applying L'Hospitals rule to the function $(\log \cos \pi/x)x^{-1}$ we conclude that $\lim_{n\to\infty} (\cos \pi/n)^n = 1$. Therefore there is a positive integer n such that $|y| > |y_n|$ where y_n is the nth vertex of the polygon P_n. It follows that there exists $0 < \beta \leqslant 1$ such that $\beta y \in P_n$. Hence there is an integer $1 \leqslant m \leqslant 2n - 1$ and a number $\gamma \geqslant 0$ such that

$$\beta y = x + x_1 + \ldots + x_{m-1} + \gamma x_m$$

We then have

$$f(y) \leqslant f(\beta y) \leqslant f(x + x_1 + \ldots + x_{m-1}) \leqslant \ldots \leqslant f(x)$$

To show f is continuous on a dense subset of X, let $|x_0| = 1$ and $S = \{\lambda x_0 : \lambda \in R\}$. Since f restricted to S is decreasing using the well-known fact that a decreasing real-valued function on R is continuous except for a countable subset of R, we conclude that f restricted to S is continuous on a dense subset $\hat{S}$ of S. Let $0 \neq x \in \hat{S}$. To show that f (not f restricted to S) is continuous at x, let x_i be a sequence converging to x. Clearly we can assume $x_i \not\perp x$. Now let $s_i = \langle x_i, x\rangle x$ and $r_i = |x_i|^2 \langle x, x_i\rangle^{-1} x$. Then $s_i, r_i \in S$ and

$$(r_i - x_i) \perp x_i, \qquad (x_i - s_i) \perp s_i, \qquad i = 1, 2, \ldots$$

Hence

$$f(r_i) = f(r_i - x_i + x_i) \leqslant f(x_i) = f(x_i - s_i + s_i) \leqslant f(s_i)$$

Since s_i, r_i both converge to x we have

$$\lim_{i\to\infty} f(r_i) = \lim_{i\to\infty} f(s_i) = f(x)$$

so $\lim_{i\to\infty} f(x_i) = f(x)$. Thus f is continuous on $\hat{S}$. It follows that f is continuous on a dense subset of X.

We say that two vectors $x, y \in X$ are *conjugate* if $\langle x,y\rangle = -1$. It is clear that 0 is not conjugate to any vector and if $x \neq 0$ then x and $-|x|^{-2}x$ are conjugate. It is also clear that x and y are conjugate if and only if $y = -|x|^{-2}x + z$ where $z \perp x$. If $\langle x,y\rangle = -1$ we call the point

$$w(x,y) = (2 + |x|^2 + |y|^2)^{-1} [(1 + |x|^2)y + (1 + |y|^2)x]$$

the *special point* for x and y. It is clear from its form that $w(x,y) = w(y,x)$, that $w(x,y)$ lies on the straight line from x to y and it is easy to check that $w(x,y) \perp w(x,y) - x$, $w(x,y) \perp w(x,y) - y$. These last two properties also characterize $w(x,y)$.

Now there are many functions f: $X \to R$ that satisfy the hypothesis of Lemma 1. For example $f(x) = \alpha(\beta + \gamma|x|^{\delta})^{-1}$, α, β, γ, $\delta > 0$ is such a function. However, the function $f(x) = (1 + |x|^2)^{-1}$ has the important further property that $f(x) + f(y) = f(w(x,y))$ whenever $\langle x,y\rangle = -1$ as is easy to check. We call a function f: $X \to R^+$ a *plane frame function* if $f(x) + f(y) = f(w(x,y))$ whenever $\langle x,y\rangle = -1$. One can check that if x, y, z are mutually conjugate then $w(x,y) = -|z|^{-2}z$. It follows that f: $X \to R^+$ is a plane frame function if and only if $f(x) + f(-|x|^{-2}x) = f(0)$ for all $x \neq 0$ and $f(x) + f(y) + f(z) = f(0)$ whenever x, y, z are mutually conjugate. We thus see that our plane frame function is a 'flattened' version of Gleason's frame function. Our main theorem shows that there is essentially only one plane frame function.

Theorem 2. If $\dim X \geqslant 2$, then f: $X \to R^+$ is a plane frame function if and only if $f(x) = f(0)(1 + |x|^2)^{-1}$.

Proof. Sufficiency has already been shown. For necessity we first show that $f(x + y) \leqslant f(y)$ if $x \perp y$. Clearly the inequality holds if $x = 0$. If $y = 0$, $x \neq 0$ then $f(x + y) = f(x) = f(0) - f(-|x|^{-2}x) \leqslant f(0) = f(y)$. Now suppose x, $y \neq 0$, $x \perp y$. Let z be the point on the straight line through y and $x + y$ that is conjugate to $x + y$. It is easily checked that $w(z, x + y) = y$ and hence $f(x + y) \leqslant f(x + y) + f(z) = f(y)$. It follows from Lemma 1 that f is continuous on a dense set in X. Suppose x and y are conjugate points and f is continuous at x. We now show f is continuous at y. Let r_i be a sequence converging to y. Since $y \neq 0$ we can assume $r_i \neq 0$, $i = 1, 2, \ldots$

Let $z=(1-\lambda)y+\lambda x$ for some $0<\lambda<1$. We can assume $r_i \neq z$, $i=1,2,\ldots$. Let

$$r_i' = (1-\lambda_i)r_i + \lambda_i z \qquad \text{and} \qquad s_i = (1-\mu_i)r_i + \mu_i z$$

where

$$\lambda_i = -(1+\langle r_i, z\rangle)(|z|^2 - \langle r_i, z\rangle^{-1} \quad \text{and} \quad \mu_i = -(1+|r_i|^2)(\langle r_i, z\rangle - |r_i|^2)^{-1}$$

Now r_i and s_i are conjugate points and z and r_i' are conjugate points. Also we see that s_i converges to x. Indeed

$$\begin{aligned}\lim_{i\to\infty} s_i &= (\langle z,y\rangle + 1)(\langle z,y\rangle - |y|^2)^{-1} y - (1+|y|^2)(\langle z,y\rangle - |y|^2) z \\ &= (\lambda - 1)\lambda^{-1} y + \lambda^{-1} z = x\end{aligned}$$

Similarly

$$\lim_{i\to\infty} r_i' = (|z|^2+1)(|z|^2 - \langle z,y\rangle)^{-1} y - (1+\langle z,y\rangle)(|z|^2 - \langle y,z\rangle)^{-1} z \equiv z_1$$

It is easy to check that z and z_1 are conjugate and since z and z_1 lie on the line through x and y we have $w(z,z_1)=w(x,y)$. It follows that

$$f(x)+f(y)=f(z_1)+f(z)$$

Also since r_i, s_i, z, r_i' are on the line through z and r_i we have

$$f(r_i)+f(s_i)=f(r_i')+f(z)=f(r_i')+f(x)+f(y)-f(z_1)$$

Hence

$$|f(r_i)-f(y)| \leqslant |f(x)-f(s_i)| + |f(r_i')-f(x)| + |f(x)-f(z_1)|$$

Let $\epsilon>0$ be given. Since f is continuous at x there is a $\delta>0$ such that $|x-y|<\delta$ implies $|f(x)-f(y)|<\epsilon$. Clearly as $\lambda\to 0$, $x\to y$ and $z_1\to x$. Let λ be such that $|x-z_1|<\delta/2$. Then for i sufficiently large

$$|r_i'-x| \leqslant |r_i'-z_1| + |z_1-x| \leqslant \delta$$

and hence $|f(r_i')-f(x)|<\epsilon$. It follows that if i is sufficiently large $|f(r_i)-f(y)|<3\epsilon$ so f is continuous at y. Now let $0\neq x_0\in X$. Since f is continuous on a dense set there is a point $0\neq x$ not in the one-dimensional subspace generated by x_0 at which f is continuous. Now there is a point z which is conjugate to both x_0 and x. By the above, f is continuous at z and again f is continuous at x_0. Now suppose $|x|=|y|\neq 0$. Let $z_i\to x$ and $|z_i|>|x|$. Then $f(y)\leqslant f(z_i)\to f|x|$ and hence $f(y)\geqslant f(x)$. Similarly $f(y)\leqslant f(x)$ so $f(y)=f(x)$ and f is only a function of distance. Define the function $g\colon R^+\to R^+$ by $g(\alpha)=f(x)$ where $|x|=\alpha$. Now g is continuous except possibly at 0 since f is. Let $\lambda>0$ and $\mu\geqslant\lambda^{-1}$ and let $|x|=\lambda$. Then there is a point y conjugate to x such that $|y|=\mu$. Since $f(x)+f(y)=f(w(x,y))$ we have

$$g(\lambda)+g(\mu)=g(|w(x,y)|)=g[(\mu^2\lambda^2-1)^{1/2}(2+\lambda^2+\mu^2)^{-1/2}]$$

Making a change of variables $\alpha=(1+\lambda^2)^{-1}$, $\beta=(1+\mu^2)^{-1}$ and introducing the function $h(u)=g[(1-u)^{1/2}u^{-1/2}]$, $0<u<1$ it is straightforward to compute that $h(\alpha) + h(\beta) = h(\alpha + \beta)$ for $0 < \alpha, \beta < 1/2$. Since h is continuous it follows that $h(\alpha) = 2h(1/2)\alpha$. Hence

$$g(\lambda) = 2g(1)(1 + \lambda^2)^{-1} \qquad \text{and} \qquad f(x) = g(|x|) = 2f(z)(1 + |x|^2)^{-1}$$

where $|z| = 1$ and $x \neq 0$. But $2f(z) = f(z) + f(-z) = f(0)$ so

$$f(x) = f(0)(1 + |x|^2)^{-1} \qquad \text{for all } x$$

It is interesting to note that Theorem 2 does not hold in a one-dimensional space. Indeed let $f(\lambda)$ be any positive function on R^+ satisfying $f(\lambda) \leqslant f(0)$ for all $\lambda \in R^+$. Define g by $g(\lambda) = f(\lambda)$ if $\lambda \geqslant 0$ and $g(\lambda) = f(0) - f(-\lambda^{-1})$ for $\lambda < 0$. Then g is a frame function on R which clearly need not be of the required form.

Let H be a Hilbert space and let $P(H)$ denote the set of orthogonal projections on H. A *state* is a map $m: P(H) \to [0,1]$ satisfying

(1) $$m(I) = 1$$

(2) $$m\left(\sum_{i=1}^{\infty} P_i\right) = \sum_{i=1}^{\infty} m(P_i)$$

if the P_i's are mutually orthogonal and the first sum is in the strong operator topology. We denote the projection onto the one-dimensional subspace generated by a vector $\phi \neq 0$ by P_ϕ. A state m is *pure* if there is a P_ϕ such that $m(P_\phi) = 1$. We now prove Gleason's theorem for pure states.

Corollary 3. Let H be a real separable Hilbert space of dimension $\geqslant 3$ and let m be a pure state satisfying $m(P_\phi) = 1$ where $|\phi| = 1$. If $P \in P(H)$ then $m(P) = \langle P\phi, \phi\rangle$.

Proof. If $\psi \perp \phi$ then $1 \geqslant m(P_\psi + P_\phi) = m(P_\psi) + 1$ so $m(P_\psi) = 0$. Now assume $0 \neq \psi \not\perp \phi$. Let $0 \neq \phi_1 \perp \phi$ so that ϕ_1, ϕ generate the two-dimensional subspace spanned by ϕ and ψ and let ϕ_2 be orthogonal to ϕ_1 and ϕ so that ϕ, ϕ_1, ϕ_2 span a three-dimensional subspace M. Let P_1 be the plane in M satisfying $P_1 = \{x \in M : \langle x, \phi\rangle = 1\}$. Now the plane P_1 can be thought of as a two-dimensional real inner product space with origin ϕ and inner product

$$\langle x, y\rangle_1 = \langle x, \phi_1\rangle\langle y, \phi_1\rangle + \langle x, \phi_2\rangle\langle y, \phi_2\rangle$$

Define a function $f: P_1 \to R^+$ by $f(x) = m(P_x)$. If x and y are conjugate points in P_1 then

$$-1 = \langle x, y\rangle_1 = \langle x, \phi_1\rangle\langle y, \phi_1\rangle + \langle x, \phi_2\rangle\langle y, \phi_2\rangle$$

so

$$\langle x, y\rangle = \langle x, \phi\rangle\langle y, \phi\rangle + \langle x, \phi_1\rangle\langle y, \phi_1\rangle + \langle x, \phi_2\rangle\langle y, \phi_2\rangle = 0$$

and $x \perp y$. Now $z = -\langle y, w(x,y)\rangle\langle x, w(x,y)\rangle^{-1}x + y$ is a vector in the subspace generated by x and y which is orthogonal to $w(x,y)$. We now show $z \perp \phi$. Indeed,

$$\langle z, \phi\rangle = -\langle z, w(x,y)\rangle\langle x, w(x,y)\rangle^{-1} + 1$$

But

$$\langle y, w(x,y)\rangle = (2 + |x|_1{}^2 + |y|_1{}^2)^{-1}(1 + |x|_1{}^2)|y|^2$$
$$= (2 + |x|_1{}^2 + |y|_1{}^2)^{-1}|x|^2|y|^2$$

and

$$\langle x, w(x,y)\rangle = (2 + |x|_1{}^2 + |y|_1{}^2)^{-1}(1 + |y|_1{}^2)|x|^2$$
$$= (2 + |x|_1{}^2 + |y|_1{}^2)^{-1}|x|^2|y|^2$$

We thus have

$$f(x) + f(y) = m(P_x) + m(P_y) = m(P_x + P_y) = m(P_{w(x,y)} + P_z)$$
$$= m(P_{w(x,y)}) + m(P_z) = m(P_{w(x,y)}) = f(w(x,y))$$

so f is a plane frame function. It follows from Theorem 2 that

$$m(P_\psi) = f(\psi) = f(0)(1 + |\psi|_1{}^2)^{-1} = m(P_\phi)(1 + |\psi|_1{}^2)^{-1}$$
$$= |\psi|^{-2} = |\psi|^{-2}\langle\psi, \phi\rangle = \langle P_\psi \phi, \phi\rangle$$

Finally if $P \in P(H)$ then there exist mutually orthogonal one-dimensional projections P_i such that $P = \sum_{i=1}^{\infty} P_i$ so

$$m(P) = \sum m(P_i) = \sum \langle P_i \phi, \phi\rangle = \langle \sum P_i \phi, \phi\rangle = \langle P\phi, \phi\rangle$$

One can easily see from examples that Corollary 3 does not hold for two-dimensional spaces. Although Gleason's full theorem does not hold in a nonseparable Hilbert space it is interesting that Corollary 3 does generalize to such spaces if we make a slightly more general definition for states. If P_α is a collection of projections we define $\vee P_\alpha$ to be the projection onto the smallest closed subspace containing the ranges of all the P_α's. If P_α is a collection of mutually orthogonal projections on a Hilbert space and m a state, since $m(I) = 1$ we have $m(P_\alpha) = 0$ except for countably many α's. We can therefore generalize the definition of state as follows. We say a map $m: P(H) \to [0, 1]$ is a *state* if

(1) $$m(I) = 1$$

(2) $$m(\vee P_\alpha) = \sum m(P_\alpha)$$

whenever P_α are mutually disjoint. With this definition of state Corollary 3 generalizes to nonseparable Hilbert spaces. Indeed, if P is a projection, let ϕ_α be an orthonormal basis for the range of P so we obtain

$$m(P) = m(\vee P_{\phi_\alpha}) = \sum m(P_{\phi_i}) = \sum \langle P_{\phi_i}\phi, \phi\rangle = \langle \sum P_{\phi_i}\phi, \phi\rangle = \langle P\phi, \phi\rangle$$

since $\phi \perp \phi_\alpha$ for $\alpha \neq 1, 2, \ldots$ where $\phi_1, \phi_2, \ldots$ are the vectors satisfying $m(P\phi_i) \neq 0$.

One should note that although Corollary 3 is proved for real spaces it is an easy step to generalize the proof so that the Corollary holds for complex or even quaternionic spaces (Varadarajan, 1968). Also notice that the full generality of Lemma 1 and Theorem 2 are not needed since for the proof of Corollary 3 all one uses is the two-dimensional space P_1.

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